Coherent Structures in the $(2+1)$-Dimensional Nonlinear Schrödinger Equation with Time-Varying Coefficients

Zheng-Yi Ma[a,b]

[a] Department of Mathematics, Lishui University, Zhejiang Lishui 323000, P.R. China
[b] Shanghai Institute of Applied Mathematics and Mechanics, Shanghai University, Shanghai 200072, P.R. China

Reprint requests to Z.-Y. M.; E-mail: mazhengyi_77@yahoo.com.cn

Z. Naturforsch. **66a,** 500 – 506 (2011) / DOI: 10.5560/ZNA.2011-0007
Received December 27, 2010 / revised April 17, 2011

Analytical solutions in terms of rational-like functions are derived for the $(2+1)$-dimensional nonlinear Schrödinger equation with time-varying coefficients using the similarity transformation and direct ansatz. By applying specific time-modulated nonlinearities, dispersions, and potentials, the dynamics of the solutions can be controlled. As a result, abundant wave structures are exhibited through chosen three types of elementary functions.

Key words: Nonlinear Schrödinger Equation; Similarity Transformation; Rational-Like Solution; Coherent Structure.
PACS numbers: 03.65.Ge; 05.45.Yv

1. Introduction

Recently, soliton theory, one of typical topics in nonlinear science, has been widely applied in optics of nonlinear media [1], photonics [2], plasmas [3], mean-field theory of Bose–Einstein condensates [4], condensed matter physics [5], and many other fields. Among them, for describing nonlinear physical phenomenon, the nonlinear Schrödinger (NLS) equation is a fundamental model. However, the coherent structures in the higher-dimensional NLS equation are seldom involved so far. In this paper, several coherent excitations of the $(2+1)$-dimensional NLS equation with time-varying coefficients are presented through the similarity transformation.

2. Explicit Solutions through Similarity Transformation

To illustrate the above idea, we focus on the $(2+1)$-dimensional nonautonomous NLS equation

$$\begin{aligned}&\mathrm{i}\frac{\partial}{\partial t}\Psi+\frac{\alpha(t)}{2}(\partial_x^2+\partial_y^2)\Psi-\frac{\Omega(t)}{2}(x^2+y^2)\Psi\\&-g(t)|\Psi|^2\Psi+\mathrm{i}\gamma(t)\Psi=0,\end{aligned}\tag{1}$$

where $\Psi\equiv\Psi(x,y,t)$ is the complex envelope of the electrical field, while $\alpha(t)$, $\Omega(t)$, and $g(t)$ represent the dispersion, the potential, and the nonlinearity coefficients, respectively, and $\gamma(t)$ the gain ($\gamma>0$) or the loss ($\gamma<0$) coefficient [6].

We first construct the following transformation for the envelope field Ψ:

$$\Psi=(\Psi_\mathrm{R}+\mathrm{i}\Psi_\mathrm{I})\mathrm{e}^{\mathrm{i}\varphi},\tag{2}$$

where the real functions $\Psi_\mathrm{R}\equiv\Psi_\mathrm{R}(x,y,t),\Psi_\mathrm{I}\equiv\Psi_\mathrm{I}(x,y,t)$, and $\varphi\equiv\varphi(x,y,t)$ [7 – 10].

For the real functions $\Psi_\mathrm{R},\Psi_\mathrm{I}$, and the phase φ, utilizing the similarity transformation, we obtain

$$\begin{aligned}&\Psi_\mathrm{R}=A+BP(\eta,\tau),\\&\Psi_\mathrm{I}=E+FQ(\eta,\tau),\\&\varphi=\chi+\mu\tau,\end{aligned}\tag{3}$$

(where the new variables $A\equiv A(t),B\equiv B(t),E\equiv E(t),F\equiv F(t),\tau\equiv\tau(t),\eta\equiv\eta(x,y,t),\chi\equiv\chi(x,y,t)$ and μ is a constant) to (2), and setting the real part and the imaginary part of (1) to zero, that is

$$\begin{aligned}&\alpha(t)B\Delta_1P_\eta-2F\Delta_2Q_\eta-(A+BP)\Delta_3\\&-[2F_t+(\alpha(t)(\chi_{xx}+\chi_{yy})+2\gamma(t)+4g(t)AE)F]Q\\&-[2E_t+(\alpha(t)(\chi_{xx}+\chi_{yy})+2\gamma(t)+2g(t)AE)E]\\&-2[\mu(A+BP)+FQ_\tau]\tau_t\\&+\alpha(t)B(\eta_x^2+\eta_y^2)P_{\eta\eta}-2g(t)[(A+BP)^3\\&+AF^2Q^2+BP(E+FQ)^2]=0,\end{aligned}\tag{4}$$

$$\begin{aligned}&\alpha(t)F\Delta_1 Q_\eta+2B\Delta_2 P_\eta-(E+FQ)\Delta_3\\&+[2B_t+(\alpha(t)(\chi_{xx}+\chi_{yy})+2\gamma(t)-4g(t)AE)B]P\\&+[2A_t+(\alpha(t)(\chi_{xx}+\chi_{yy})+2\gamma(t))A]\\&+2(BP_\tau-\mu(E+F))\tau_t+\alpha(t)(\eta_x^2+\eta_y^2)FQ_{\eta\eta}\\&-2g(t)[FQ(F^2Q^2+(A+BP)^2)\\&+E(A^2+B^2P^2+E^2+EFQ+3F^2Q^2)]=0.\end{aligned}\tag{5}$$

Here, $\Delta_i (i=1,2,3)$ is a symbol of the expression

$$\Delta_1=\eta_{xx}+\eta_{yy},\tag{6}$$
$$\Delta_2=\eta_t+\alpha(t)(\chi_x\eta_x+\chi_y\eta_y),\tag{7}$$
$$\Delta_3=2\chi_t+\alpha(t)(\chi_x^2+\chi_y^2)+\Omega(t)(x^2+y^2).\tag{8}$$

The reduction equations $\Delta_i (i=1,2,3)=0$ and $2\sigma_t+[\alpha(t)(\chi_{xx}+\chi_{yy})+2\gamma(t)]\sigma=0 (\sigma=A,B,F)$ can deduce

$$\begin{aligned}&\eta=\delta_1(t)x+\delta_2(t)y+\delta_0(t),\\&\Omega(t)=-\frac{2\chi_t+\alpha(t)(\chi_x^2+\chi_y^2)}{x^2+y^2},\end{aligned}\tag{9}$$

$$\begin{aligned}\chi=-\frac{1}{\alpha(t)}\Bigg[&\frac{\delta_1(t)\delta_1(t)_t-\delta_2(t)\delta_2(t)_t}{2\delta_1(t)^2}x^2\\&+\frac{\delta_2(t)_t y+\delta_0(t)}{\delta_1(t)}x\Bigg]+d_0(t),\end{aligned}\tag{10}$$

$$\begin{aligned}&A=a_0\mathrm{e}^{\frac{1}{2}\int_0^t\frac{\delta_1(s)\delta_1(s)_s-\delta_2(s)\delta_2(s)_s-2\delta_1(s)^2\gamma(s)}{\delta_1(s)^2}\mathrm{d}s},\\&B=bA, F=fA,\end{aligned}\tag{11}$$

where a_0, b, and f are arbitrary constants, $\delta_0(t)$, $\delta_1(t)$, $\delta_2(t)$, and $d_0(t)$ are functions of time t, $E=0$. When taking the variable τ and nonlinearity $g(t)$ as

$$\begin{aligned}&\tau=\frac{1}{2}\int_0^t\alpha(s)(\delta_1(s)^2+\delta_2(s)^2)\mathrm{d}s,\\&g(t)=-\frac{\alpha(t)(\delta_1(t)^2+\delta_2(t)^2)}{2A^2},\end{aligned}\tag{12}$$

the coupled system of constant coefficients from (4) and (5) are reduced to

$$\begin{aligned}&(\mu-1)fQ-bP_\tau-fQ(f^2Q^2+b^2P^2+2bP)\\&-fQ_{\eta\eta}=0,\end{aligned}\tag{13}$$

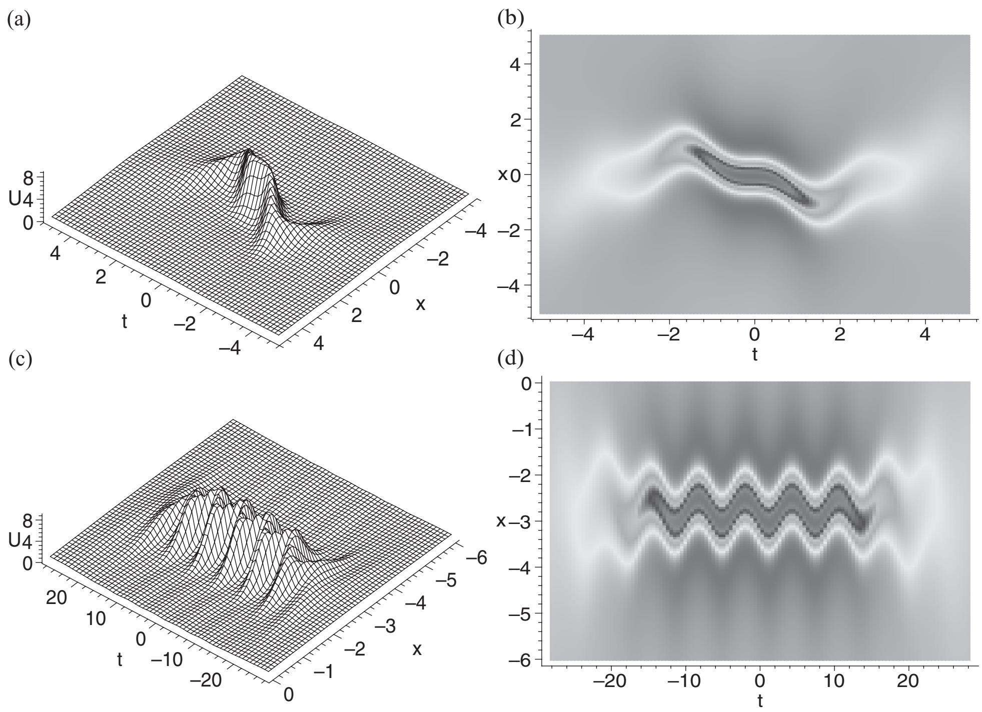

Fig. 1. (a) – (b) Solitary structure with a snaking behaviour and the contour plot for the intensity $U\equiv|\Psi|^2$ (19) of the first-order rational-like solution (18) for $a_0=\delta_1(t)=1, \delta_0(t)=\sin^3(t), \alpha(t)=0.5\tanh(t)$, and $\gamma(t)=0.1\tanh(t)\mathrm{sech}(t)$. (c) – (d) Coherent structure with chirp and the contour plot for the intensity $U\equiv|\Psi|^2$ (19) of the first-order rational-like solution (18) for $a_0=\delta_1(t)=1, \delta_0(t)=\mathrm{e}^{1+0.1\sin(t)}, \alpha(t)=\sin^2(0.02t)$, and $\gamma(t)=\sin^3(0.005t)$.

$$(1+bP)(\mu-(f^2Q^2+(1+bP)^2) + fQ_\tau - bP_{\eta\eta} = 0. \quad (14)$$

Therefore, the first-order rational solution of (13) and (14) is

$$P = -\frac{4}{R_1(\eta,\tau)b}, \quad Q = -\frac{8\tau}{R_1(\eta,\tau)f}, \quad (15)$$

where $R_1(\eta,\tau) = 1+2\eta^2+4\tau^2$, and the second-order rational solution is

$$P = \frac{P_1(\eta,\tau)}{R_2(\eta,\tau)b}, \quad Q = -\frac{Q_1(\eta,\tau)\tau}{R_2(\eta,\tau)f}, \quad (16)$$

where $P_1(\eta,\tau) = \frac{3}{8} - 9\tau^2 - \frac{3}{2}\eta^2 - 6\eta^2\tau^2 - 10\tau^4 - \frac{1}{2}\eta^4$, $Q_1(\eta,\tau) = -\frac{15}{4} + 2\tau^2 - 3\eta^2 + 4\eta^2\tau^2 + 4\tau^4 + \eta^4$, $R_2(\eta,\tau) = \frac{3}{32} + \frac{33}{8}\tau^2 + \frac{9}{16}\eta^2 - \frac{3}{2}\eta^2\tau^2 + \frac{9}{2}\tau^4 + \frac{1}{8}\eta^4 + \frac{2}{3}\tau^6 + \eta^2\tau^4 + \frac{1}{2}\eta^4\tau^2 + \frac{1}{12}\eta^6$ according to the direct method developed in [8, 11, 12] ($\mu = 1$).

Finally, a direct reduction solution of (1) can be driven

$$\Psi = A(1+bP+\mathrm{i}fQ)\mathrm{e}^{\mathrm{i}(\chi+\tau)}, \quad (17)$$

the known functions $A \equiv A(t), P \equiv P(\eta,\tau), Q \equiv Q(\eta,\tau), \chi \equiv \chi(x,y,t)$, and $\tau \equiv \tau(t)$ are expressed by (11), (15), (16), (10), and (12), respectively.

3. Several Coherent Excitations of the Solutions

Now, we focus on the coherent structures of the solution for the (2 + 1)-dimensional nonautonomous NLS equation (1).

According to (17), the first-order rational-like solution of (1) can be rewritten as

$$\Psi = a_0\left(\frac{-3+2\eta^2+4\tau^2-8\mathrm{i}\tau}{1+2\eta^2+4\tau^2}\right) \cdot \mathrm{e}^{\frac{1}{2}\int_0^t \frac{\delta_1(s)\delta_1(s)_s - \delta_2(s)\delta_2(s)_s - 2\delta_1(s)^2\gamma(s)}{\delta_1(s)^2}\mathrm{d}s}\mathrm{e}^{\mathrm{i}(\chi+\tau)}, \quad (18)$$

the variables η, τ, and χ are expressed by (9), (12), and (10), respectively. We can find that the above solution (18) is just a restricted combination of the coefficients $\alpha(t)$, $\Omega(t)$, $g(t)$, and $\gamma(t)$. As the time t func-

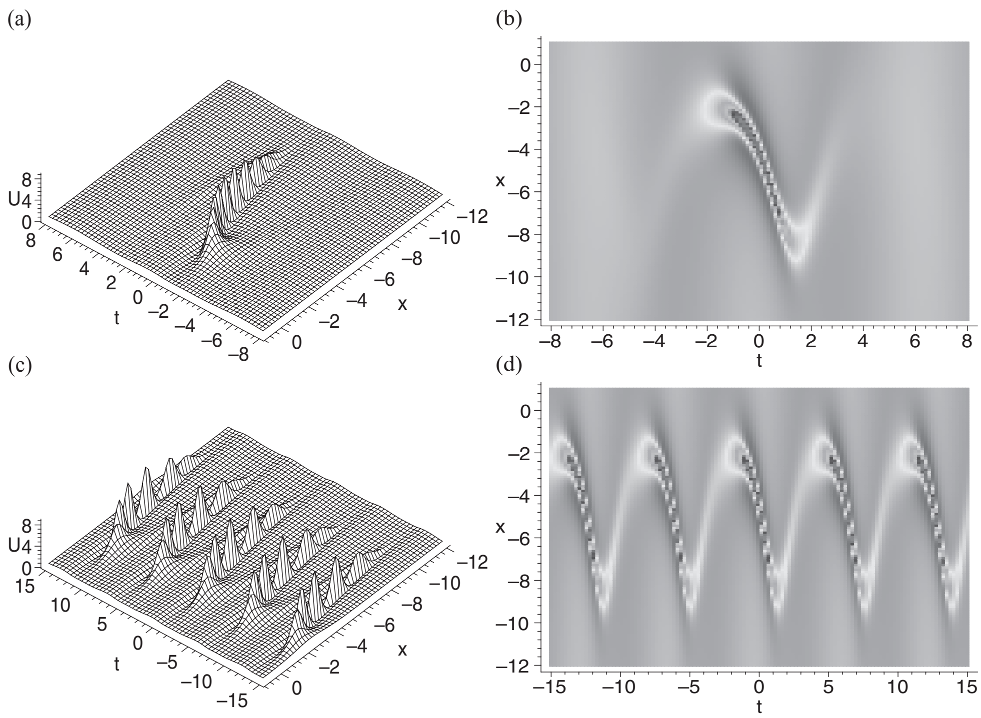

Fig. 2. (a) – (b) Single chirped structure with N-shape and the contour plot for the intensity $U \equiv |\Psi|^2$ (19) of the first-order rational-like solution (18) for $a_0 = \delta_1(t) = 1, \delta_0(t) = 1+\mathrm{e}^{1+\sin(t)}, \alpha(t) = \tanh(t)$, and $\gamma(t) = 0.1\sin(t)$. (c) – (d) Periodic structure with chirp and the contour plot for the intensity $U \equiv |\Psi|^2$ (19) for $a_0 = \delta_1(t) = 1, \delta_0(t) = 1+\mathrm{e}^{1+\sin(t)}, \alpha(t) = \tanh(\sin(t))$, and $\gamma(t) = 0.1\sin(t)$.

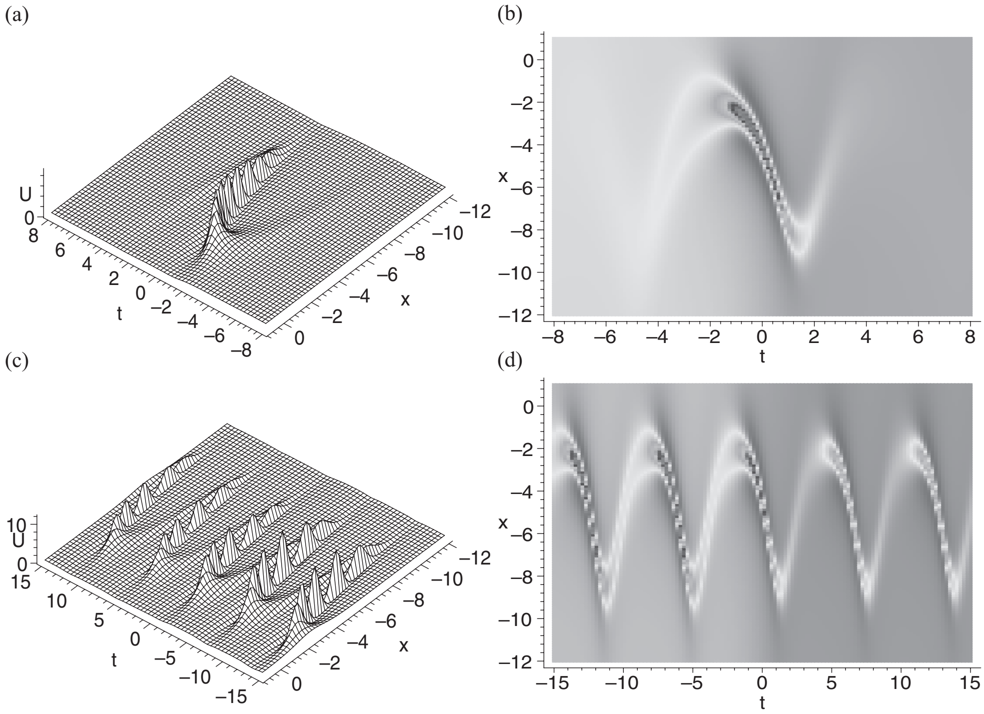

Fig. 3. (a) – (b) Another chirped structure with N-shape and the contour plot for the intensity $U \equiv |\Psi|^2$ (19) of the first-order rational-like solution (18) for $a_0 = \delta_1(t) = 1, \delta_0(t) = 1 + e^{1+\sin(t)}, \alpha(t) = \tanh(t)$, and $\gamma(t) = 0.1\mathrm{sech}(t)$. (c) – (d) Quasi-periodic structure with chirp and the contour plot for the intensity $U \equiv |\Psi|^2$ (19) of the first-order rational-like solution (18) for $a_0 = \delta_1(t) = 1, \delta_0(t) = 1 + e^{1+\sin(t)}, \alpha(t) = \tanh(\sin(t))$, and $\gamma(t) = 0.1\mathrm{sech}(t)$.

tions $\delta_0(t), \delta_1(t), \delta_2(t), d_0(t), \gamma(t)$, and $\alpha(t)$ are arbitrary, several typical time-modulated excitations of the intensity

$$U \equiv |\Psi|^2 = a_0^2 e^{\int_0^t \frac{\delta_1(s)\delta_1(s)_s - \delta_2(s)\delta_2(s)_s - 2\delta_1(s)^2\gamma(s)}{\delta_1(s)^2} ds} \cdot \frac{[-3 + 2(\delta_1(t)x + \delta_2(t)y + \delta_0(t))^2 + 4\tau^2]^2 + 64\tau^2}{[1 + 2(\delta_1(t)x + \delta_2(t)y + \delta_0(t))^2 + 4\tau^2]^2}, \tag{19}$$

where $\tau = \frac{1}{2}\int_0^t \alpha(s)(\delta_1(s)^2 + \delta_2(s)^2) ds$, are derived.

It can be seen that the solution structure (18) is different from the solution of the generalized (3 + 1)-dimensional Gross–Pitaevskii (GP) equation (20) or (21) [13], where by applying a novel similarity transformation, the (3 + 1)-dimensional GP equation is reduced to a (3 + 1)-dimensional standard NLS equation, and the solution of the GP equation is thus constructed via those of the NLS equation. For simplification, we fix the parameters $a_0 = \delta_1(t) = 1$. When taking $\delta_0(t) = \sin^3(t), \alpha(t) = 0.5\tanh(t)$, and $\gamma(t) = 0.1\tanh(t)\mathrm{sech}(t)$, a solitary structure with the snaking behaviour in a symmetric time interval $[-5, 5]$ shows twisting variation on its propagation for the fixed $y = 0$ (Fig. 1a – b). Another case, when taking $\delta_0(t) = e^{1+0.1\sin(t)}, \alpha(t) = \sin^2(0.02t)$, and $\gamma(t) = \sin^3(0.005t)$, a coherent structure with chirp in a symmetric time interval $[-28, 28]$ shows oscillated variation for the time-t and space-x depended solution (Fig. 1c – d).

Now, we focus on the periodic propagating wave pattern in terms of some elementary functions for (19). Considering solution (18), for analytical convenience, we restrict the attention to simple cases where the quantities $\delta_0(t), \alpha(t)$, and $\gamma(t)$ can be evaluated in simple, closed forms which do not involve the complicated integrals in this paper. The selections of the triangle function sin, the hyperbolic function tanh/sech, and the exponential function will satisfy these requirements, and will now lead to new wave patterns for the (2+1)-dimensional nonautonomous NLS equation (1). Figure 2a – b describes a single chirped structure with N-shape and the contour plot for the intensity $U \equiv |\Psi|^2$

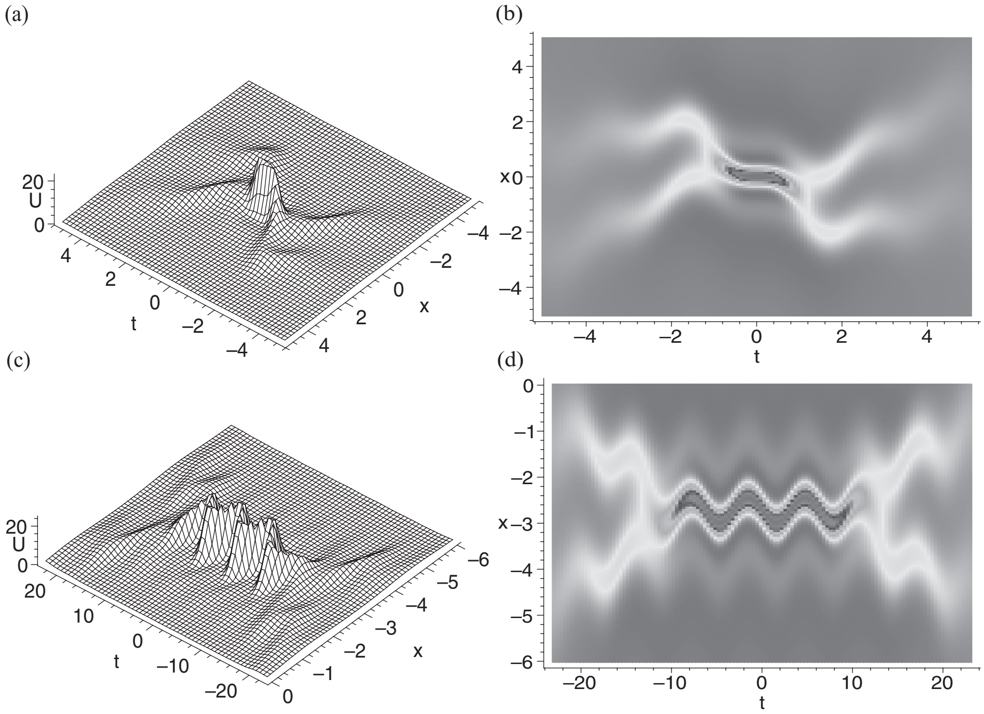

Fig. 4. Double structures of the intensity (21), for functions selected corresponding to Figure 1.

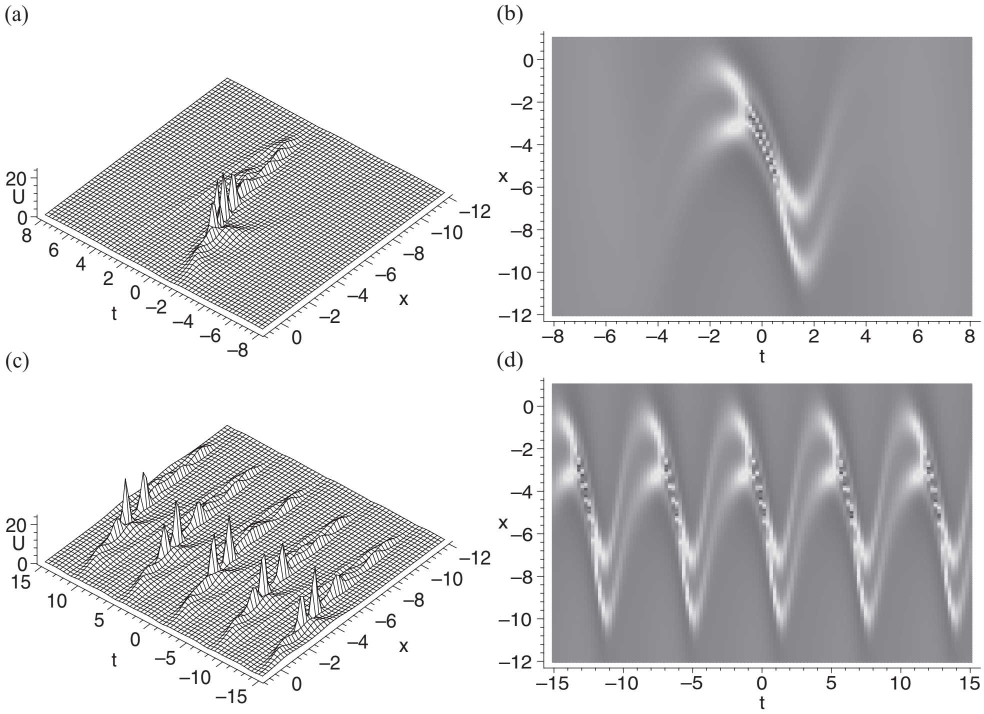

Fig. 5. Double structures of the intensity (21), for functions selected corresponding to Figure 2.

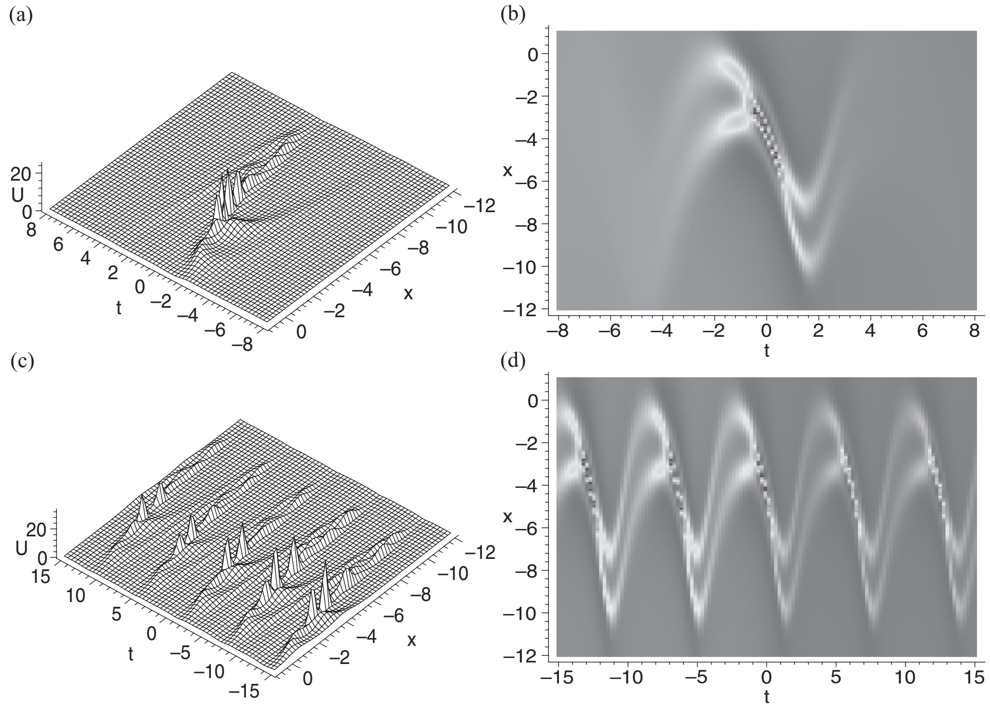

Fig. 6. Double structures of the intensity (21), for functions selected corresponding to Figure 3.

(19) of the first-order rational-like solution (18) (the fixed $y=0$), for $\delta_0(t)=1+\mathrm{e}^{1+\sin(t)}, \alpha(t)=\tanh(t)$, and $\gamma(t)=0.1\sin(t)$. Figure 2c – d are the periodic structure with chirp and the contour plot for the intensity $U \equiv |\Psi|^2$ (19) when the dispersion coefficient $\alpha(t)=\tanh(\sin(t))$.

The corresponding circumstance of Figure 2 is the quasi-periodic propagation. When the coefficient $\gamma(t)=0.1\sin(t)$ is substituted by the hyperbolic function $\gamma(t)=0.1\mathrm{sech}(t)$, the quasi-periodic structure with chirp for the intensity $U \equiv |\Psi|^2$ (19) of the first-order rational-like solution (18) is constructed (Fig. 3a – b). Figure 3c – d depicts the pattern behaviour in a symmetric time interval $[-15,\ 15]$, where the chirping amplitude becomes weak as time t increases from -15 to 15. Obviously, all these structures are different from our former works [14 – 17].

The second-order rational-like solution of (1) can be rewritten as

$$\Psi = a_0\left(1+\frac{P_1(\eta,\tau)}{R_2(\eta,\tau)}-\frac{Q_1(\eta,\tau)\tau\mathrm{i}}{R_2(\eta,\tau)}\right) \cdot \mathrm{e}^{\frac{1}{2}\int_0^t \frac{\delta_1(s)\delta_1(s)_s-\delta_2(s)\delta_2(s)_s-2\delta_1(s)^2\gamma(s)}{\delta_1(s)^2}\mathrm{d}s}\,\mathrm{e}^{\mathrm{i}(\chi+\tau)}, \tag{20}$$

its intensity

$$U \equiv |\Psi|^2 = a_0^2\mathrm{e}^{\int_0^t \frac{\delta_1(s)\delta_1(s)_s-\delta_2(s)\delta_2(s)_s-2\delta_1(s)^2\gamma(s)}{\delta_1(s)^2}\mathrm{d}s} \cdot \frac{(P_1(\eta,\tau)+R_2(\eta,\tau))^2+(Q_1(\eta,\tau)\tau)^2}{(R_2(\eta,\tau))^2}, \tag{21}$$

here $P_1(\eta,\tau)=\frac{3}{8}-9\tau^2-\frac{3}{2}\eta^2-6\eta^2\tau^2-10\tau^4-\frac{1}{2}\eta^4$, $Q_1(\eta,\tau)=-\frac{15}{4}+2\tau^2-3\eta^2+4\eta^2\tau^2+4\tau^4+\eta^4$, $R_2(\eta,\tau)=\frac{3}{32}+\frac{33}{8}\tau^2+\frac{9}{16}\eta^2-\frac{3}{2}\eta^2\tau^2+\frac{9}{2}\tau^4+\frac{1}{8}\eta^4+\frac{2}{3}\tau^6+\eta^2\tau^4+\frac{1}{2}\eta^4\tau^2+\frac{1}{12}\eta^6$, also, the variables η,τ and χ are expressed by (9), (12), and (10), respectively, the time t functions $\delta_0(t),\delta_1(t),\delta_2(t),d_0(t),\gamma(t)$, and $\alpha(t)$ are arbitrary, and $\tau=\frac{1}{2}\int\alpha(s)(\delta_1(s)^2+\delta_2(s)^2)\mathrm{d}s$. Figures 4 – 6 show: (i) Although the same functions taken as in Figures 1 – 3, these excitations have double structures comparing to the former single ones. (ii) The second-order rational-like solution owns even higher amplitude, just as the description of rogue waves in the deep ocean and high intensity rogue light wave pulses in optical fibers [11].

4. Conclusion

In summary, we have obtained analytical solutions in terms of rational-like functions for the $(2+1)$-dimensional nonlinear Schrödinger equation with time-varying coefficients using the similarity transformation and direct ansatz. These obtained solutions contain several free functions of time t, which provide us with more chooses of these functions to generate the abundant wave structures. Here, we chose three types of elementary functions to exhibit these wave propagations related to the obtained solutions. These solutions may provide more information to further study the nonlinear physical system.

Acknowledgement

The author expresses his sincere thanks for the referees for their valuable suggestions and is in debt to Profs. J.F. Zhang and C.L. Zheng and Drs. Y. Yang and Y.M. Chen for their fruitful discussions. The work was supported by the National Natural Science Foundation of China (No. 10772110) and the Natural Science Foundation of Zhejiang Province of China (Nos. Y606049 and Y6090681).

[1] Y. Kivshar and G. P. Agrawal, Optical Solitons: From Fibers to Photonic Crystals, Academic Press, New York 2003.
[2] A. Hasegawa, Optical Solitons in Fibers, Springer-Verlag, Berlin 1989.
[3] R. K. Dodd, J. C. Eilbeck, J. D. Gibbon, and H. C. Morris, Solitons and nonlinear wave equations, Academic Press, New York 1982.
[4] L. P. Pitaevskii and S. Stringari, Bose–Einstein Condensation,Oxford University Press, Oxford 2003.
[5] A. Scott, Nonlinear Science: Emergence and Dynamics of Coherent Structures, Oxford Appl. and Eng. Mathematics, Vol. 1, Oxford 1999.
[6] arXiv:nlin/0512061v1 [nlin.PS] 21 Dec 2005.
[7] Z. Y. Yan, Phys. Scr. **75**, 320 (2007).
[8] Z. Y. Yan, Phys. Lett. A **374**, 672 (2010).
[9] Z. Y. Yan, Constructive Theory and Applications of Complex Nonlinear Waves, Science Press, Beijing 2007.
[10] Z. Y. Yan and V. V. Konotop, Phys. Rev. E **80**, 036607 (2009).
[11] N. Akhmediev, J. M. Soto-Crespo, and A. Ankiewicz, Phys. Rev. E **80**, 026601 (2009).
[12] N. Akhmediev, A. Ankiewicz, and M. Taki, Phys. Lett. A **373**, 675 (2009).
[13] Y. Gao and S. Y. Lou, Commun. Theor. Phys. (Beijing, China) **52**, 1031 (2009).
[14] Y. H. Hu and Z. Y. Ma, Chaos, Solitons, Fractals **34**, 482 (2007).
[15] W. H. Huang, Y. L. Liu, and Z. Y. Ma, Commun. Theor. Phys. (Beijing, China) **47**, 397 (2007).
[16] X. F. Wu, Z. Y. Ma, and J. M. Zhu, Z. Naturforsch. **62a**, 1 (2007).
[17] Z. Y. Ma, Appl. Math. Comput. **194**, 67 (2007).